TRUNCATION IN HAHN FIELDS

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1. Introduction

Let $\boldsymbol{k}$ be a field and Γ an (additive) ordered[1] abelian group. Then we have the Hahn field $K := \boldsymbol{k}((t^{\Gamma}))$ whose elements are the formal series $a = \sum_{\gamma} a_{\gamma} t^{\gamma}$ with coefficients $a_{\gamma} \in \boldsymbol{k}$ and exponents γ ranging over Γ, and having well-ordered support

$$\operatorname{supp} a \ := \ \{\gamma \in \Gamma : \ a_{\gamma} \neq 0\}.$$

These series are added and multiplied as suggested by the series notation, with $t^{\alpha} \cdot t^{\beta} = t^{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$, making K a field with $\boldsymbol{k}$ as a subfield via the identification $c = ct^0$ for $c \in \boldsymbol{k}$. We assume familiarity with basic facts of this nature.

Given such a series $a = \sum_{\gamma} a_{\gamma} t^{\gamma} \in K$ and $\alpha \in \Gamma$ we can truncate the series at t^{α} to give the series $a|_{\alpha} := \sum_{\gamma < \alpha} a_{\gamma} t^{\gamma} \in K$. We call $a|_{\alpha}$ a *truncation* of a; if in addition $a \neq a|_{\alpha}$ (equivalently, $\alpha \leqslant \beta$ for some $\beta \in \operatorname{supp} a$), then $a|_{\alpha}$ is said to be a *proper truncation* of a. Thus all truncations of a are proper iff $\operatorname{supp} a$ is cofinal in Γ. The only proper truncation of ct^{α} with nonzero $c \in \boldsymbol{k}$ is 0.

A subset of K is said to be *closed under truncation* (or just *truncation closed*) if it contains all truncations of all its elements. The property of being truncation closed turns out to be stable under various operations. I learned about this from Ressayre [10] (where $\boldsymbol{k} = \mathbb{R}$) and confess to being very surprised by this phenomenon. My modest goal here is to bring together in one place the various stability results of this kind, with proofs and without unnecessary restrictions. For example, the next theorem (from Section 2) gives the stability of truncation closedness under certain arithmetic extension procedures.

Theorem 1.1. *If A is a truncation closed subset of K, then the ring generated by A in K and the field generated by A in K are truncation closed.*

Let $v \colon K^{\times} \to \Gamma$ be the usual valuation on K, that is, $va = \min \operatorname{supp} a$ for nonzero $a \in K$. By convention we extend v to all of K by $v0 := \infty > \Gamma$. Let

$$\mathcal{O} \ = \ \{a \in K : \ va \geqslant 0\}$$

be the valuation ring of v, with maximal ideal $\mathfrak{m} := \{a \in K : \ v(a) > 0\}$.

Let F be a valued field, that is, a field (also called F) together with a valuation ring $\mathcal{O}_F$ of F. We commit the customary abuse of language by calling the valued field F *henselian* if the local ring $\mathcal{O}_F$ is henselian. We consider K as a (henselian) valued field by taking $\mathcal{O}$ as its distinguished valuation ring. Let E be a subfield of K. We consider E as a valued field by taking $\mathcal{O}_E := \mathcal{O} \cap E$ as its valuation ring. Note that there is a smallest (with respect to inclusion) subfield of K that

[1] As usual, "ordered" in "ordered group" and "ordered field" refers to a *total* ordering on the underlying set that is compatible with the group operation, respectively the ring operations.

is henselian and contains E, the *henselization* E^{h} of E in K (whose valuation ring $\mathcal{O}_{E^{\mathrm{h}}}$ is a henselization of the local ring $\mathcal{O}_E$). In Sections 4 and 5 we prove:

Theorem 1.2. *Let $E \supseteq \boldsymbol{k}$ be a truncation closed subfield of K. Then:*

(1) *the henselization E^{h} of E in K is truncation closed;*
(2) *the relative algebraic closure of E in K is truncation closed.*

The proof of (1) in Section 4 is "characteristic free", but that of (2) in Section 5 requires extra work in positive characteristic. As an example of (2), consider the case that $\boldsymbol{k} = \mathbb{R}$ and Γ is divisible, so K is real closed. Let E be a truncation closed subfield of K. Then E as an ordered subfield of K has a real closure inside K, and (2) says that this real closure of E in K is again truncation closed. In this way we recover a result in [9].

Hahn fields have an appealing universality property: if F is a henselian valued field whose value group Γ has a copy in F, and residue field $\boldsymbol{k}$ with $\operatorname{char} \boldsymbol{k} = 0$, then there exists a copy of F in the Hahn field $\boldsymbol{k}((t^{\Gamma}))$; see Kaplansky [6]. In Section 6 we explain the meaning of "copy" and show that with the same assumptions on F there exists even a *truncation closed* copy of F in $\boldsymbol{k}((t^{\Gamma}))$.

For $A \subseteq K$ we let $A^{\times n}$ be the set of n-tuples $\vec{a} = (a_1, \dots, a_n)$ with $a_1, \dots, a_n \in A$. Let $X_1, \dots, X_n$ be distinct indeterminates. Then we have the formal power series ring $\boldsymbol{k}[[X_1, \dots, X_n]]$. In Section 4 we define for any n-tuple $\vec{a} = (a_1, \dots, a_n) \in \mathfrak{m}^{\times n}$ a $\boldsymbol{k}$-algebra morphism

$$f(X) \mapsto f(\vec{a}) : \ \boldsymbol{k}[[X_1, \dots, X_n]] \to \mathcal{O}.$$

In Section 7 we consider adjunctions of the following kind. Let for each $n \geqslant 1$ a subset $\mathcal{F}_n$ of $\boldsymbol{k}[[X_1, \dots, X_n]]$ be given such that the subring $\boldsymbol{k}[X_1, \dots, X_n, \mathcal{F}_n]$ of $\boldsymbol{k}[[X_1, \dots, X_n]]$ is closed under $\partial/\partial X_i$ for $i = 1, \dots, n$, and let $\mathcal{F}$ be the family $(\mathcal{F}_n)$. For example, if $\operatorname{char}(\boldsymbol{k}) = 0$, we could take

$$\mathcal{F}_1 = \{(1 + X_1)^{-1},\ \exp X_1,\ \log(1 + X_1)\}, \qquad \mathcal{F}_n = \emptyset \ \text{ for } n > 1$$

where $\exp X_1 := \sum_{i=0}^{\infty} X_1^i / i!$ and $\log(1 + X_1) := \sum_{i=1}^{\infty} (-1)^{i+1} X_1^i / i$. A subfield E of K is said to be $\mathcal{F}$-closed if $f(\vec{a}) \in E$ for all $f \in \mathcal{F}_n$ and $\vec{a} \in (\mathfrak{m} \cap E)^{\times n}$, $n = 1, 2, \dots$. The $\mathcal{F}$-closure of a subfield E of K is the smallest $\mathcal{F}$-closed subfield $\mathcal{F}(E)$ of K that contains E.

Theorem 1.3. *If $\operatorname{char}(\boldsymbol{k}) = 0$ and $E \supseteq \boldsymbol{k}$ is a truncation closed subfield of K, then its $\mathcal{F}$-closure $\mathcal{F}(E)$ is also truncation closed.*

We extend some of these results in two ways: one is to allow K to be a *twisted Hahn field* $\boldsymbol{k}((t^{\Gamma}; \mathfrak{c}))$ where $\mathfrak{c} : \Gamma \times \Gamma \to \boldsymbol{k}^{\times}$ is a *factor system*: for all $\alpha, \beta, \gamma \in \Gamma$,

$$\mathfrak{c}(\alpha, \beta) \cdot \mathfrak{c}(\alpha + \beta, \gamma) = \mathfrak{c}(\alpha, \beta + \gamma) \cdot \mathfrak{c}(\beta, \gamma), \qquad \mathfrak{c}(\alpha, \beta) = \mathfrak{c}(\beta, \alpha), \qquad \mathfrak{c}(0, \alpha) = 1.$$

The field $\boldsymbol{k}((t^{\Gamma}; \mathfrak{c}))$ is defined just as $\boldsymbol{k}((t^{\Gamma}))$, except that $t^{\alpha} \cdot t^{\beta} = \mathfrak{c}(\alpha, \beta) t^{\alpha+\beta}$, so $1/t^{\alpha} = \mathfrak{c}(\alpha, -\alpha)^{-1} t^{-\alpha}$. The field $\boldsymbol{k}$ is identified with a subring of $\boldsymbol{k}((t^{\Gamma}; \mathfrak{c}))$ by $c = ct^0$ for $c \in \boldsymbol{k}$. If $\mathfrak{c}(\alpha, \beta) = 1$ for all α, β, we get the usual Hahn field $\boldsymbol{k}((t^{\Gamma}))$.

The other way is to allow $\boldsymbol{k}$ to be a division ring and Γ to be an ordered group which is not necessarily commutative (while keeping the additive notation for Γ). Then $\boldsymbol{k}((t^{\Gamma}))$ is a division ring, with multiplication according to $ct^{\alpha} \cdot dt^{\beta} = cdt^{\alpha+\beta}$ for $c, d \in \boldsymbol{k}^{\times}$ and $\alpha, \beta \in \Gamma$, and we identify the division ring $\boldsymbol{k}$ with a subring of $\boldsymbol{k}((t^{\Gamma}))$ as before. We then refer to $\boldsymbol{k}((t^{\Gamma}))$ as a *Neumann division ring*[2].

[2]Neumann [7] even allows twisting in this noncommutative set-up, but we omit this.

For such a twisted Hahn field or Neumann division ring K the notions involving truncation are defined as before, and so is the map $v : K \to \Gamma \cup \{\infty\}$, the subring $\mathcal{O}$ of K, and the two-sided ideal $\mathfrak{m}$ of $\mathcal{O}$, although we shall refrain from calling v a valuation or $\mathcal{O}$ a valuation ring when K is not commutative. In any case, $\boldsymbol{k} \subseteq \mathcal{O}$ maps isomorphically onto the residue ring $\mathcal{O}/\mathfrak{m}$ under the residue map $\mathcal{O} \to \mathcal{O}/\mathfrak{m}$.

Here are some notational conventions applying to Hahn fields and Neumann division rings $K = \boldsymbol{k}((t^\Gamma))$, as well as to twisted Hahn fields $K = \boldsymbol{k}((t^\Gamma; \mathfrak{c}))$. We say that the sum $\sum_i a_i$ of a (possibly infinite) family $(a_i)_{i \in I}$ of elements of K *exists* if the following conditions are satisfied:

(i) for each $\gamma \in \Gamma$ there are only finitely many $i \in I$ with $\gamma \in \operatorname{supp} a_i$;
(ii) the union of the supports of the a_i is well-ordered.

It is clear which element of K is then denoted by $\sum_i a_i$ (or just $\sum a_i$).

We let a, b, c range over elements of K, and α, β, γ over Γ. When using expressions like $a = \sum a_\gamma t^\gamma$ it is understood that the sum ranges over all γ, and that $a_\gamma \in \boldsymbol{k}$ for all γ. Clearly, $(a+b)|_\alpha = a|_\alpha + b|_\alpha$, and if (a_i) is a family in K for which $\sum a_i$ exists, then $\sum a_i|_\alpha$ exists and $(\sum a_i)|_\alpha = \sum a_i|_\alpha$.

To take full advantage of the possibility of arguments by (transfinite) induction, we define $o(a)$ to be the ordinal that is order isomorphic to $\operatorname{supp} a$. We let $E^\times = E \setminus \{0\}$ be the multiplicative group of a field (or even a division ring) E. Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \dots\}$.

This paper is meant as a survey of known results and gives, I hope, efficient and transparent proofs. While writing it I thought that the positive characteristic case of Theorem 1.2 might be new, but the referee pointed out Fornasiero [5], which contains this theorem in full generality, as well as most other results in this paper. In Section 8 we say more on the origin of the material, and mention some applications.

2. Stability under arithmetic adjunctions

In this section A and B are subsets of our Hahn field $K = \boldsymbol{k}((t^\Gamma))$. We set

$$A + B := \{a + b : a \in A,\ b \in B\}, \quad A - B := \{a - b : a \in A,\ b \in B\}.$$

In the next (obvious) lemma, we do not assume that A or B is truncation closed.

Lemma 2.1. *If a has all its truncations in A and b has all its truncations in B, then $a + b$ has all its truncations in $A + B$.*

Note that we did not assume $a \in A$ or $b \in B$, but by considering elements $a \in A$ and $b \in B$ we conclude that if A and B are truncation closed, then so is $A + B$ (and $A - B$, by taking $-B$ instead of B).

The *additive monoid* generated by A in K is by definition the smallest subset $\Sigma(A)$ of K that contains $A \cup \{0\}$ and is closed under addition; thus

$$\Sigma(A) = \{a_1 + \cdots + a_n : a_1, \dots, a_n \in A,\ n = 1, 2, 3, \dots\} \cup \{0\}.$$

Corollary 2.2. *If A is truncation closed, then so is $\Sigma(A)$.*

Proof. Assume $A \subseteq K$ is truncation closed. The set $\Sigma(A)$ is the union of the sets $\{0\}, A, A + A, A + A + A, \dots$, each of which is truncation closed by the above. Hence $\Sigma(A)$ is truncation closed. □

In particular, if A is truncation closed, then so is the additive subgroup $\Sigma(A-A)$ of K generated by A. If A is an additive subgroup of K and $\Gamma \neq \{0\}$ and all truncations of $a = \sum a_\gamma t^\gamma$ are in A, then each term $a_\gamma t^\gamma$ is a difference of two truncations of a and so is in A as well. Next we prove an analogue of Lemma 2.1 for products, and we set $AB := \{ab : a \in A, b \in B\}$.

Lemma 2.3. *Let A be an additive subgroup of K. Suppose a has all its truncations in A and b has all its truncations in B. Then ab has all its truncations in $\Sigma(AB)$.*

Proof. By induction on $(o(a), o(b))$ with the lexicographic ordering on the relevant set of pairs of ordinals. If $o(a) = 0$ or $o(b) = 0$, then $a = 0$ or $b = 0$, and we are done. Let $o(a) > 0$ and $o(b) > 0$. We want to show that $ab|_\alpha$ belongs to $\Sigma(AB)$. Assume first that there are $\sigma \in \operatorname{supp} a$ and $\tau \in \operatorname{supp} b$ with $\sigma + \tau \geqslant \alpha$. Then

$$a = a_1 + a_2 \quad \text{with} \quad a_1 := a|_\sigma = \sum_{\gamma<\sigma} a_\gamma t^\gamma, \quad a_2 := \sum_{\gamma \geqslant \sigma} a_\gamma t^\gamma,$$

$$b = b_1 + b_2 \quad \text{with} \quad b_1 := b|_\tau = \sum_{\gamma<\tau} b_\gamma t^\gamma, \quad b_2 := \sum_{\gamma \geqslant \tau} b_\gamma t^\gamma,$$

and A is an additive subgroup of K, so all truncations of a_1 and a_2 lie in A, and all truncations of b_1 lie in B. Now $ab = a_1 b + a_2 b_1 + a_2 b_2$, hence

$$(ab)|_\alpha = (a_1 b)|_\alpha + (a_2 b_1)|_\alpha.$$

Since $o(a_1) < o(a)$ and $o(a_2) \leqslant o(a)$, $o(b_1) < o(b)$, we can assume inductively that $(a_1 b)|_\alpha$ and $(a_2 b_1)|_\alpha$ belong to $\Sigma(AB)$, so $(ab)|_\alpha$ is in $\Sigma(AB)$. If for all $\sigma \in \operatorname{supp} a$ and $\tau \in \operatorname{supp} b$ we have $\sigma + \tau < \alpha$, then $\operatorname{supp} a$ and $\operatorname{supp} b$ are not cofinal in Γ, so $a \in A$, $b \in B$ and hence $(ab)|_\alpha = ab \in \Sigma(AB)$ as well. □

Corollary 2.4. *Suppose A and B are truncation closed, and A is an additive subgroup of K. Then $\Sigma(AB)$ is truncation closed.*

Proof. Let $c \in \Sigma(AB)$. Then $c = a_1 b_1 + \cdots + a_n b_n$ with $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, and so $c|_\alpha = (a_1 b_1)|_\alpha + \cdots + (a_n b_n)|_\alpha$. Now $(a_i b_i)|_\alpha$ lies in $\Sigma(AB)$ for $i = 1, \ldots, n$, by Lemma 2.3, and so $c|_\alpha$ lies in $\Sigma(AB)$ as well. □

Corollary 2.5. *Suppose A is truncation closed. Then the ring generated by A in K is truncation closed.*

Proof. The ring generated by A in K is $Z[A]$ where Z is the image of $\mathbb{Z}$ under the ring morphism $\mathbb{Z} \to \boldsymbol{k}$. Since $A \cup Z$ is truncation closed, we can arrange $Z \subseteq A$. Next, by passing to $\Sigma(A - A)$ we arrange that A is also an additive subgroup of K. Then $Z[A]$ is the increasing union of the additive subgroups $A_0, A_1, A_2, \ldots$ with $A_0 := A$, and $A_{n+1} := \Sigma(A_n A_n)$. By induction on n it follows from Corollary 2.4 that each A_n is truncation closed, and so $Z[A]$ is as well. □

Corollary 2.6. *If R is a truncation closed subring of K and all proper truncations of all $a \in A$ lie in R, then the ring $R[A]$ is truncation closed.*

Proof. Apply Corollary 2.5 to the set $R \cup A$ in the role of A. □

The next lemma does not assume that E is truncation closed or $a \in E$.

Lemma 2.7. *Let E be a subfield of K containing all truncations of $a \in \mathfrak{m}$. Then all truncations of $\sum a^n = (1-a)^{-1}$ lie in E.*

Proof. By induction on $o(a)$. Let $\alpha > 0$, so $a = a_1 + a_2$ where $\operatorname{supp} a_1$ consists of all $\gamma \in \operatorname{supp} a$ with $n\gamma < \alpha$ for all n, while $\operatorname{supp} a_2$ consists of all $\gamma \in \operatorname{supp} a$ with $n\gamma \geqslant \alpha$ for some n. Then $a_1 \in E$ and all truncations of a_1 and a_2 lie in E. If $a_2 = 0$, then

$$(\sum a^n)|_\alpha \;=\; \sum a^n \;=\; \sum a_1^n \;=\; (1-a_1)^{-1} \in E.$$

Assume $a_2 \neq 0$, so that $o(a_1) < o(a)$. Let $\sigma := v(a_2)$ be the smallest element of $\operatorname{supp} a_2$, and take m maximal with $m\sigma < \alpha$; possibly $m = 0$. Then for $n \geqslant m$,

$$a^n \;=\; a_1^n + \cdots + \binom{n}{m} a_1^{n-m} a_2^m + \text{terms of valuation} \geqslant \alpha,$$

and this holds also for $n < m$ provided summands with negative exponent of a_1 are interpreted as 0. Hence

$$\begin{aligned}(\sum a^n)|_\alpha \;&=\; \sum_{i=0}^{m} (\sum_{n \geqslant i} \binom{n}{i} a_1^{n-i} a_2^i)|_\alpha \\ &=\; \sum_{i=0}^{m} \big((1-a_1)^{-(i+1)} a_2^i\big)|_\alpha.\end{aligned}$$

By the inductive assumption applied to a_1 and Lemma 2.3 each of the $m+1$ terms of the last sum belongs to E. Therefore $(\sum a^n)|_\alpha$ belongs to E. □

Corollary 2.8. *Suppose R is a truncation closed subring of K. Then the fraction field of R in K is also truncation closed.*

Proof. Consider the following subset of the fraction field E of R in K:

$$T(E) := \{a \in E : \text{ all truncations of } a \text{ lie in } E\}.$$

Clearly $R \subseteq T(E)$, and by its very definition $T(E)$ is truncation closed. Thus $T(E)$ is a subring of E by Corollary 2.5 applied to $A = T(E)$. Let now $b \in T(E)$, $b \neq 0$; it is enough to show that then $b^{-1} \in T(E)$. For $\beta = vb$ we have $b = ct^\beta(1-a)$ with $c \in \boldsymbol{k}^\times$ and $a \in \mathfrak{m}$. Then $ct^\beta \in T(E)$, hence $c^{-1}t^{-\beta} \in T(E)$, and thus $c^{-1}t^{-\beta}b = 1 - a \in T(E)$, so $a \in T(E)$. Then $(1-a)^{-1} \in T(E)$ by Lemma 2.7 , and so $b^{-1} = (1-a)^{-1}c^{-1}t^{-\beta} \in T(E)$. □

Note that Corollaries 2.5 and 2.8 yield Theorem 1.1.

Corollary 2.9. *If E is a truncation closed subfield of K and all proper truncations of all $a \in A$ lie in E, then the field $E(A)$ is truncation closed.*

Note that if E is a truncation closed subfield of K, then the field $\boldsymbol{k}_E := \boldsymbol{k} \cap E$ is a subring of $\mathcal{O}_E$ which is mapped isomorphically onto the residue field $\mathcal{O}_E/\mathfrak{m}_E$ of $\mathcal{O}_E$ under the residue map $\mathcal{O}_E \to \mathcal{O}_E/\mathfrak{m}_E$, where $\mathfrak{m}_E := \mathfrak{m} \cap E$.

3. Variants

Section 2 goes through for a twisted Hahn field $K = \boldsymbol{k}((t^\Gamma; \mathfrak{c}))$, with only a minor change in the proof of Corollary 2.8.

Next, let $K = \boldsymbol{k}((t^\Gamma))$ be a Neumann division ring. Then Section 2 also goes through, where $R[A]$ (a notation customary only for commutative rings) denotes the ring generated over R by A in K. Also, in Lemma 2.7, *subfield* should be replaced by *division subring*, and its proof requires a change: for example, when the m in that proof equals 2, one needs to consider $\sum_{i,j,k} a_1^i a_2 a_1^j a_2 a_1^k$ (with i, j, k ranging over $\mathbb{N}$) and to observe that it equals

$$(1-a_1)^{-1}a_2(1-a_1)^{-1}a_2(1-a_1)^{-1}$$

rather than its "commutative" value $(1-a_1)^{-3}a_2^2$. Moreover, in Corollary 2.8 the term *fraction field of R in K* should be interpreted as the division ring generated by R in K, *the field $E(A)$* in Corollary 2.9 becomes the division ring generated over E by A in K. The final remark introducing $\boldsymbol{k}_E$ now takes the following form: If E is a truncation closed division subring of K, then $\boldsymbol{k}_E := \boldsymbol{k} \cap E$ is a division subring of $\mathcal{O}_E := \mathcal{O} \cap E$, and is mapped isomorphically onto the residue ring $\mathcal{O}_E/\mathfrak{m}_E$ under the residue map $\mathcal{O}_E \to \mathcal{O}_E/\mathfrak{m}_E$, where $\mathfrak{m}_E := \mathfrak{m} \cap E$.

Another variant. Let us first address a minor pathology. Let E be a truncation closed subfield of the Hahn field $K = \boldsymbol{k}((t^\Gamma))$. If $\boldsymbol{k}_E = \boldsymbol{k}$ (that is, $E \supseteq \boldsymbol{k}$), then E is clearly contained in the subfield $\boldsymbol{k}_E((t^\Delta))$ of K, with $\Delta = v(E^\times)$. This containment does not always hold when $\boldsymbol{k}_E \neq \boldsymbol{k}$. For example, let $\boldsymbol{k} := \mathbb{R}$, $\Gamma = \mathbb{Z}$, and $r \in \mathbb{R} \setminus \mathbb{Q}$. Then we have the truncation closed subfield $E := \mathbb{Q}((rt))$ of $K = \mathbb{R}((t))$ whose elements are the Laurent series $\sum_k c_k r^k t^k$ with all $c_k \in \mathbb{Q}$. Then $\boldsymbol{k}_E = \mathbb{Q}$, but E is not contained in the subfield $\mathbb{Q}((t))$ of K.

This prompts the following definition, where now $K = \boldsymbol{k}((t^\Gamma))$ is allowed to be a Neumann division ring. Call a subset A of K *strongly truncation closed* if it is truncation closed and contains with every $a = \sum_\gamma a_\gamma t^\gamma$ also all coefficients a_γ. (If $A \supseteq \boldsymbol{k}$, this is the same as truncation closed.) Then all results of Section 2 go through upon replacing everywhere *truncation closed* by *strongly truncation closed.* For Corollary 2.8, the proof needs extra attention: given the strongly truncation closed subring R of K we need to show that the truncation closed subring $T(E)$ defined in this proof is strongly truncation closed. To see why $T(E)$ has this property, let $\boldsymbol{k}_R$ be the division subring of $\boldsymbol{k}$ generated by $R \cap \boldsymbol{k}$. Then clearly $\boldsymbol{k}_R \subseteq E$, and so $\boldsymbol{k}_R \subseteq T(E)$. Let Δ be the subgroup of Γ generated by the $\gamma \in \Gamma$ such that $ct^\gamma \in R$ for some $c \in \boldsymbol{k}^\times$. Then $R \subseteq \boldsymbol{k}_R((t^\Delta))$, so $E \subseteq \boldsymbol{k}_R((t^\Delta))$, and thus $T(E)$ is strongly truncation closed.

4. Stability under henselization

In this section we prove that truncation closedness is preserved under henselization. Throughout this section $K = \boldsymbol{k}((t^\Gamma))$ is a Hahn field.

Let $X_1, \dots, X_m$ be distinct indeterminates and put $X = (X_1, \dots, X_m)$. We let $i = (i_1, \dots, i_m)$ range over $\mathbb{N}^m$. Let $f(X) = \sum_i c_i X^i \in \boldsymbol{k}[[X]]$ with all $c_i \in \boldsymbol{k}$, and $X^i := X_1^{i_1} \cdots X_m^{i_m}$. For $\vec{a} = (a_1, \dots, a_m) \in \mathfrak{m}^{\times m}$ we put $\vec{a}^i := a_1^{i_1} \cdots a_m^{i_m}$, and we have an element $f(\vec{a}) = \sum_i c_i \vec{a}^i \in \mathcal{O}$ since the union of the supports of the $c_i \vec{a}^i$ is contained in the (well-ordered) submonoid of Γ generated by the union of the supports of $a_1, \dots, a_m$. Moreover,

$$f(X) \mapsto f(\vec{a}) \; : \; \boldsymbol{k}[[X]] \to \mathcal{O}$$

is a $\boldsymbol{k}$-algebra morphism. At this point we need some general facts about the henselization of a local ring and its behaviour under adjoining variables. We define "henselization" in the next subsection and state some of its properties there without proof. (A construction of the henselization and the proofs omitted here are for example in [12], Section 12.5.)

Henselization. We assume familiarity with the notion of a henselian local ring. (In some accounts "local ring" includes being noetherian, but not here.) Below, A and B are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. Call A a *local in B* if A is a subring of B and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$ (and thus $A \cap \mathfrak{m}_B = \mathfrak{m}_A$). A *local morphism* $\phi : A \to B$ is a ring morphism $\phi : A \to B$ such that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (and thus $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$). So if A is local in B, then the inclusion $A \to B$ is a local morphism.

The local ring A has a *henselization*: this is a local morphism $j : A \to A^{\mathrm{h}}$ into a henselian local ring A^{h} such that any local morphism $\phi : A \to B$ into a henselian local ring B equals $\psi \circ j$ for a unique local morphism $\psi : A^{\mathrm{h}} \to B$. This universal property determines $j : A \to A^{\mathrm{h}}$ up-to-unique-isomorphism over A. Moreover, $j : A \to A^{\mathrm{h}}$ is injective, and accordingly we identify A with a subring of A^{h} via j, and call A^{h} the henselization of A. So A is local in A^{h}. Here are some basic properties of the henselization A^{h}:

(i) the maximal ideal of A^{h} is the ideal $\mathfrak{m}_A A^{\mathrm{h}}$ generated by $\mathfrak{m}_A$ in A^{h};
(ii) the induced ring morphism $A/\mathfrak{m}_A \to A^{\mathrm{h}}/\mathfrak{m}_A A^{\mathrm{h}}$ is an isomorphism;
(iii) A^{h} is a faithfully flat A-algebra;
(iv) if A is an integrally closed domain, then so is A^{h};
(v) if A is an integrally closed domain and $b \in A^{\mathrm{h}}, b \neq 0$, then $A \cap bA^{\mathrm{h}} \neq \{0\}$.

Suppose A is an integrally closed domain, and A is local in the henselian local ring B. Then the inclusion $A \to B$ extends uniquely to a local morphism $A^{\mathrm{h}} \to B$, and by (v), this map $A^{\mathrm{h}} \to B$ is injective. Its image in B we denote by A^{h} as well, calling it *the henselization of A in B* to avoid confusion; it is local in B.

We now drop the assumption that A is an integrally closed domain. Let $X = (X_i)$ be a family of distinct indeterminates. Then we have the polynomial ring $A[X] \supseteq A$. The ideal $(\mathfrak{m}_A, X)$ generated by $\mathfrak{m}_A$ and the X_i in $A[X]$ is a maximal ideal, and so we have the localization $A[X]_{(\mathfrak{m}_A, X)} = S^{-1}A[X]$ with $S = 1 + (\mathfrak{m}_A, X)$. The natural map $A[X] \to S^{-1}A[X]$ is injective, and we consider $A[X]$ as a subring of $S^{-1}A[X]$ via this map. Then A is local in $S^{-1}A[X]$. We let $A[X;\mathrm{h}]$ be the henselization of $S^{-1}A[X]$. The maximal ideal of $A[X;\mathrm{h}]$ is generated by $\mathfrak{m}_A$ and the X_i, and for any local morphism $\phi : A \to B$ into a henselian local ring B and any family (b_i) in $\mathfrak{m}_B$, there is a unique extension of ϕ to a ring morphism $A[X;\mathrm{h}] \to B$ sending X_i to b_i, for all $i \in I$. (This ring morphism is automatically local.) Let $Y = (Y_j)$ be a second family of distinct indeterminates, distinct also from the X_i. Then

$$A[X,Y] \;=\; A[X]\,[Y] \;\subseteq\; A[X;\mathrm{h}]\,[Y] \;\subseteq\; A[X;\mathrm{h}]\,[Y;\mathrm{h}]$$

shows how $A[X,Y]$ is to be seen as a subring of $A[X;\mathrm{h}]\,[Y;\mathrm{h}]$. We now have

$$A[X,Y;\mathrm{h}] \;=\; A[X;\mathrm{h}]\,[Y;\mathrm{h}]$$

in the sense that the $A[X,Y]$-algebra $A[X;\mathrm{h}]\,[Y;\mathrm{h}]$ has the universal property that characterizes the $A[X,Y]$-algebra $A[X,Y;\mathrm{h}]$ up-to-unique-isomorphism.

Suppose A is an integrally closed henselian local domain and $X = (X_1, \dots, X_m)$. The formal power series ring $A[[X]]$ is then a henselian local domain by [11]. We regard $A[X]$ and $S^{-1}A[X]$ as subrings of $A[[X]]$ in the obvious way. Then $S^{-1}A[X]$ is an integrally closed domain and local in $A[[X]]$. Thus the henselization $A[X;\mathrm{h}]$ of $S^{-1}A[X]$ in $A[[X]]$ is a subring of $A[[X]]$. With $Y = (Y_1, \dots, Y_n)$ a new tuple of distinct variables, $A[X,Y;\mathrm{h}]$ is likewise a subring of $A[[X,Y]]$. Since $A[X;\mathrm{h}]$ is integrally closed by (iv), $A[X;\mathrm{h}]\,[Y;\mathrm{h}]$ is likewise a subring of $A[X;\mathrm{h}]\,[[Y]]$.

A correct reading of "$A[X,Y;\mathrm{h}] = A[X;\mathrm{h}]\ [Y;\mathrm{h}]$" now gives the following: if $f(X,Y) \in A[X,Y;\mathrm{h}]$ and the $f_j(X) \in A[[X]]$ for $j \in \mathbb{N}^n$ are such that

$$f(X,Y) \ = \ \sum_{j \in \mathbb{N}^n} f_j(X) Y^j \in A[[X]]\ [[Y]] \ = \ A[[X,Y]],$$

then $f_j(X) \in A[X;\mathrm{h}]$ for all j. This is used below in the proof of Lemma 4.2.

Back to truncation. Let $X = (X_1, \dots, X_m)$ as before. We now apply the above to $f(\vec{a})$ with $f \in \boldsymbol{k}[X;\mathrm{h}]$ and $\vec{a} \in (E \cap \mathfrak{m})^{\times m}$ where E is a subfield of $K = \boldsymbol{k}((t^\Gamma))$. Let $\mathcal{O}_E^{\mathrm{h}}$ be the henselization of $\mathcal{O}_E$ in $\mathcal{O}$. Recall from the introduction that (abusing notation) E^{h} denotes the fraction field of $\mathcal{O}_E^{\mathrm{h}}$ in K, so $\mathcal{O}_{E^h} = \mathcal{O}_E^{\mathrm{h}}$.

Lemma 4.1. *Let $E \supseteq \boldsymbol{k}$ be a subfield of K, let $a_1, \dots, a_m \in E \cap \mathfrak{m}$, and set $\vec{a} := (a_1, \dots, a_m)$. Then $f(\vec{a}) \in E^{\mathrm{h}}$ for all $f(X) \in \boldsymbol{k}[X;\mathrm{h}]$.*

Proof. By the universal property of $\boldsymbol{k}[X;\mathrm{h}]$, the substitution map

$$f(X) \mapsto f(\vec{a}) : \boldsymbol{k}[X] \to \mathcal{O}_{E^{\mathrm{h}}}$$

extends uniquely to a local morphism $\boldsymbol{k}[X;\mathrm{h}] \to \mathcal{O}_{E^{\mathrm{h}}}$. By composition with the inclusion $\mathcal{O}_{E^{\mathrm{h}}} \to \mathcal{O}$, this gives the unique local morphism $\boldsymbol{k}[X;\mathrm{h}] \to \mathcal{O}$ extending the substitution map $f(X) \mapsto f(\vec{a}) : \boldsymbol{k}[X] \to \mathcal{O}$. It remains to note that this local morphism $\boldsymbol{k}[X;\mathrm{h}] \to \mathcal{O}$ coincides with the restriction to $\boldsymbol{k}[X;\mathrm{h}]$ of the substitution map $f(X) \mapsto f(\vec{a}) : \boldsymbol{k}[[X]] \to \mathcal{O}$, again by the universal property of $\boldsymbol{k}[X;\mathrm{h}]$. □

Lemma 4.2. *Let $E \supseteq \boldsymbol{k}$ be a subfield of K, let $a_1, \dots, a_m \in E \cap \mathfrak{m}$ and suppose E contains all truncations of $a_1, \dots, a_m$; set $\vec{a} := (a_1, \dots, a_m)$. Let $f(X) \in \boldsymbol{k}[X;\mathrm{h}]$. Then all truncations of $f(\vec{a})$ lie in E^{h}.*

Proof. By induction on $\big(o(a_1), \dots, o(a_m)\big)$, with the lexicographic ordering on the relevant set of m-tuples of ordinals. If $o(a_k) = 0$ for $k = 1, \dots, m$, then $f(\vec{a}) = f(0) \in \boldsymbol{k}$, and we are done.

Assume next that $o(a_k) \neq 0$ for some $k \in \{1, \dots, m\}$. Let $\alpha \in \Gamma^{>0}$; then for $k = 1, \dots, m$ we have $a_k = b_k + c_k$ where the support of $b_k \in \mathfrak{m}$ consists of those $\gamma \in \operatorname{supp} a_k$ with $n\gamma < \alpha$ for all n, while the support of $c_k \in \mathfrak{m}$ consists of those $\gamma \in \operatorname{supp} a_k$ with $n\gamma \geqslant \alpha$ for some n. Note that b_k is a truncation of a_k, so $b_k \in E$ for all k. Set

$$\vec{b} := (b_1, \dots, b_m), \qquad \vec{c} := (c_1, \dots, c_m).$$

If all c_k are 0, then $f(\vec{a})|_\alpha = f(\vec{a}) = f(\vec{b}) \in E^{\mathrm{h}}$. Suppose $c_k \neq 0$ for some k. Then $o(b_k) < o(a_k)$ for some k, and $o(b_k) \leqslant o(a_k)$ for all k. Take $r \in \mathbb{N}$ so large that $rv(c_k) \geqslant \alpha$ for all k. Let $Y_1, \dots, Y_m$ be new indeterminates and set $Y = (Y_1, \dots, Y_m)$. Then $f(X+Y) := f(X_1 + Y_1, \dots, X_m + Y_m) \in \boldsymbol{k}[X,Y;\mathrm{h}]$, and

$$f(X+Y) \ = \ \sum_i f_i(X) Y^i, \qquad \text{all } f_i(X) \in \boldsymbol{k}[X;\mathrm{h}],$$

with $i = (i_1, \dots, i_m)$ ranging over $\mathbb{N}^m$. Set $|i| = i_1 + \cdots + i_m$ for such i. Then

$$f(\vec{a}) \ = \ f(\vec{b} + \vec{c}) \ = \ \sum f_i(\vec{b}) \vec{c}^{\,i} \ = \ \sum_{|i|<r} f_i(\vec{b}) \vec{c}^{\,i} + d \ \text{ with } v(d) \geqslant \alpha.$$

Thus $f(\vec{a})|_\alpha = \big(\sum_{|i|<r} f_i(\vec{b}) \vec{c}^{\,i}\big)|_\alpha$. We can assume inductively that all truncations of all $f_i(\vec{b})$ are in E^{h}. Moreover, E contains all truncations of $c_1, \dots, c_m$, hence E^{h} contains $f(\vec{a})|_\alpha$, by Lemmas 2.1 and 2.3. □

Theorem 4.3. *Let E be a strongly truncation closed subfield of K. Then E^{h} is strongly truncation closed.*

Proof. Let $\Delta = v(E^{\times}) \subseteq \Gamma$ be the value group of E. Then $\boldsymbol{k}_E((t^{\Delta}))$ is a henselian subfield of K. Now E is strongly truncation closed, so $E \subseteq \boldsymbol{k}_E((t^{\Delta}))$, and thus $E^{\mathrm{h}} \subseteq \boldsymbol{k}_E((t^{\Delta}))$. So we can replace K by $\boldsymbol{k}_E((t^{\Delta}))$ and arrange in this way that $\boldsymbol{k} = \boldsymbol{k}_E$ and $\Delta = \Gamma$. The idea is now to use the fact that E^{h} is generated over E by elements of the form $f(\vec{a})$ with $f \in \boldsymbol{k}[X; \mathrm{h}]$ and $\vec{a} \in (\mathfrak{m} \cap E)^{\times m}$, and then apply Lemma 4.2. Here are the details. Let $m \geqslant 1$ and put

$$p_m(X,T) \ := \ 1 + T + X_1 T^2 + \cdots + X_m T^{m+1} \in \boldsymbol{k}[X,T].$$

Since the local ring $\boldsymbol{k}[X; \mathrm{h}]$ is henselian, the polynomial $p_m(X,T) \in \boldsymbol{k}[X; \mathrm{h}][T]$ has a unique zero $f_m = f_m(X)$ in $\boldsymbol{k}[X; \mathrm{h}]$. Let $\vec{a} \in (\mathfrak{m} \cap E)^{\times m}$. Then $f_m(\vec{a})$ must be the unique zero of the polynomial $p_m(\vec{a}, T) \in \mathcal{O}_E[T]$ in the henselian local ring $\mathcal{O}_{E^{\mathrm{h}}}$. In fact, E^{h} is generated as a field over E by its elements $f_m(\vec{a})$ with $m = 1, 2, 3, \ldots$ and $\vec{a} \in (\mathfrak{m} \cap E)^{\times m}$; see for example section 12.5 in [12]. Thus E^{h} is truncation closed by Lemma 4.2 and Corollary 2.8. □

Suppose $\mathrm{char}(\boldsymbol{k}) = 0$ and Γ has rank > 1. Then there is a truncation closed subfield $E \supseteq \boldsymbol{k}$ of K with a quadratic extension $F \subseteq E^{\mathrm{h}}$ that is not truncation closed: Take elements $\alpha, \beta > 0$ such that $n\alpha < \beta$ for all n. Set $E := \boldsymbol{k}(t^{\alpha}, t^{\beta}) \subseteq K$, so E is truncation closed and t^{α} and t^{β} are algebraically independent over $\boldsymbol{k}$. Let

$$F \ := \ E(\sqrt{1 + t^{\alpha} + t^{\beta}}) \ \subseteq \ K,$$

where $\sqrt{1 + t^{\alpha} + t^{\beta}}$ is given by the usual binomial expansion. Thus $F \subseteq E^{\mathrm{h}}$, and $\sqrt{1 + t^{\alpha}}$ is a truncation of $\sqrt{1 + t^{\alpha} + t^{\beta}}$, but $\sqrt{1 + t^{\alpha}} \notin F$.

5. Stability for algebraic extensions

In this section $K = \boldsymbol{k}((t^{\Gamma}))$ is a Hahn field, although at the end we indicate what goes through for *twisted* Hahn fields. We do not assume $E \supseteq \boldsymbol{k}$ in the next result.

Theorem 5.1. *Suppose* $\mathrm{char}(\boldsymbol{k}) = 0$ *and E is a henselian truncation closed subfield of K. Then any algebraic field extension $F \subseteq K$ of E is truncation closed.*

Proof. Let $F \subseteq K$ be an algebraic field extension of E. To show that F is truncation closed, we can assume that $[F : E] < \infty$, and by further straightforward reductions familiar from the theory of algebraic extensions of henselian valued fields of residue characteristic zero, it suffices to prove the result in two cases:

Case that $F|E$ is unramified, that is, $v(F^{\times}) = v(E^{\times})$. Then $[\boldsymbol{k}_F : \boldsymbol{k}_E] = [F : E]$, and so $F = E(\boldsymbol{k}_F)$ is truncation closed by Corollary 2.9.

Case that $F = E(b)$ where $b^p \in E^{\times}$ with p a prime number, and $vb \notin v(E^{\times})$. Then $b^p = rt^{\gamma}(1 + a)$ with $r \in \boldsymbol{k}^{\times},\ a \in E,\ va > 0$. Since E is henselian, we have $1 + a = (1 + c)^p$ with $c \in E,\ vc > 0$, and so, replacing b by $b(1 + c)^{-1}$ we arrange $b = rt^{\gamma}$, which has 0 as its only proper truncation. Thus $F = E(b)$ is truncation closed by Corollary 2.9. □

By the example at the end of Section 4 we cannot drop here the assumption that E is henselian. An example later in this section shows that the assumption $\mathrm{char}(\boldsymbol{k}) = 0$ cannot be omitted either in Theorem 5.1.

We now turn to the positive characteristic variant of Theorem 5.1 where F is the relative algebraic closure of E in K. This requires some lemmas about Artin-Schreier extensions.

Artin-Schreier extensions. In this subsection we assume: $\operatorname{char}(\boldsymbol{k}) = p > 0$, $\boldsymbol{k}$ is perfect, and $p\Gamma = \Gamma$. It follows that then K is perfect: for $a = \sum a_\gamma t^\gamma$ we have $a^{1/p} = \sum a_\gamma^{1/p} t^{\gamma/p}$. In particular, $\operatorname{supp} a^{1/p} = (1/p) \operatorname{supp} a$. Note that if E is a truncation closed subfield of K, then so is $E^{1/p}$.

Artin-Schreier extensions that do not cause a corresponding proper extension of the value group often cause trouble in valuation theory. Lemmas 5.2 and 5.3 deal with such extensions.

Lemma 5.2. *Suppose that* $\operatorname{supp} a < 0$. *Then* $\sum_{n=1}^\infty a^{1/p^n}$ *exists.*

Proof. Obviously $\bigcup_{n=1}^\infty \operatorname{supp} a^{1/p^n} < 0$. Let $\gamma < 0$ be given. Then $\gamma \in \operatorname{supp} a^{1/p^n}$ for only finitely many $n \geqslant 1$: otherwise, $p^n\gamma \in \operatorname{supp} a$ for infinitely many $n \geqslant 1$, contradicting that $\operatorname{supp} a$ is well-ordered. Also, $\bigcup_{n=1}^\infty \operatorname{supp} a^{1/p^n}$ is well-ordered: Otherwise we have $1 \leqslant n_1 < n_2 < n_3 < \cdots$ and $\gamma_i \in \operatorname{supp} a^{1/p^{n_i}}$ for $i = 1, 2, 3, \ldots$ such that $\gamma_1 > \gamma_2 > \gamma_3 > \cdots$, and so $p^{n_i}\gamma_i \in \operatorname{supp} a$ for all i, and $p^{n_1}\gamma_1 > p^{n_2}\gamma_2 > p^{n_3}\gamma_3 > \cdots$, contradicting again that $\operatorname{supp} a$ is well-ordered. □

For a such that $\operatorname{supp} a < 0$, set $\omega(a) := \sum_{n=1}^\infty a^{1/p^n}$; thus $\omega(a)$ is a solution of the Artin-Schreier equation $x^p - x = a$.

Lemma 5.3. *Let E be a perfect truncation closed subfield of K and let $a \in E$ be such that* $\operatorname{supp} a < 0$. *Suppose that $\omega(b) \in E$ for every proper truncation b of a. Then all proper truncations of $\omega(a)$ lie in E.*

Proof. Let $\alpha \in \operatorname{supp}\ \omega(a)$ be such that $\omega(a)|_\alpha \neq \omega(a)$; our job is to show that then $\omega(a)|_\alpha \in E$. Clearly $\alpha \in (1/p^m) \operatorname{supp} a < 0$ for some $m \geqslant 1$, and so $\alpha < 0$. For archimedean Γ we don't need the assumption that $\omega(b) \in E$ for every proper truncation b of a : in this case we can take $n \geqslant 1$ so high that $\alpha \leqslant (1/p^n) \operatorname{supp} a$, which gives $\omega(a)|_\alpha = \sum_{i=1}^{n-1} a^{1/p^i} \in E$. In the general case, we have $a = a_1 + a_2$ where $\operatorname{supp} a_1$ consists of the $\gamma \in \operatorname{supp} a$ with $n\gamma < \alpha$ for all $n \geqslant 1$, and $\operatorname{supp} a_2$ consists of the $\gamma \in \operatorname{supp} a$ with $n\gamma \geqslant \alpha$ for some $n \geqslant 1$. Then a_1 is a proper truncation of a and $\omega(a) = \omega(a_1) + \omega(a_2)$ with $\operatorname{supp} \omega(a_1) < \operatorname{supp} \omega(a_2)$. So we can replace a by a_2, and in this way arrange that for every $\gamma \in \operatorname{supp} a$ there is $n \geqslant 1$ with $n\gamma \geqslant \alpha$. Then the same argument as for archimedean Γ applies. □

A variant of Theorem 5.1 in positive characteristic. For a subfield E of K we let E^{a} be the relative algebraic closure of E in K.

Theorem 5.4. *Suppose* $\operatorname{char}(k) = p > 0$ *and* $E \supseteq \boldsymbol{k}$ *is a truncation closed subfield of K. Then E^{a} is truncation closed.*

Proof. We begin with reducing to the case where K is algebraically closed and E is henselian with divisible value group, and perfect. Let $\boldsymbol{k}^{\mathrm{ac}}$ be an algebraic closure of the field $\boldsymbol{k}$ and let $\mathbb{Q}\Gamma$ be the (ordered) divisible hull of Γ. Then K is a truncation closed subfield of the Hahn field $\boldsymbol{k}^{\mathrm{ac}}((t^{\mathbb{Q}\Gamma}))$. Also, $E(\boldsymbol{k}^{\mathrm{ac}})$ is a truncation closed subfield of $\boldsymbol{k}^{\mathrm{ac}}((t^{\mathbb{Q}\Gamma}))$ by Corollary 2.9, and is algebraic over E. Since E^{a} is the intersection of K with the relative algebraic closure of $E(\boldsymbol{k}^{\mathrm{ac}})$ in $\boldsymbol{k}^{\mathrm{ac}}((t^{\mathbb{Q}\Gamma}))$, we may replace K by $\boldsymbol{k}^{\mathrm{ac}}((t^{\mathbb{Q}\Gamma}))$ and E by $E(\boldsymbol{k}^{\mathrm{ac}})$, and in this way reduce to the case that $\boldsymbol{k}$ and K are algebraically closed, which is assumed below.

Since E^{a} contains the henselization E^{h} of E in K and E^{h} is truncation closed, we can also assume that E is henselian. Let $\Delta := v(E^{\times})$ be the value group of E, so $E \subseteq \boldsymbol{k}((t^{\Delta})) \subseteq K$, and E contains each t^{δ} with $\delta \in \Delta$. Then each t^{γ} with $\gamma \in \mathbb{Q}\Delta$ is algebraic over E, and so the subfield $E(t^{\gamma} : \gamma \in \mathbb{Q}\Delta)$ of $\boldsymbol{k}((t^{\mathbb{Q}\Delta})) \subseteq K$ is algebraic over E, and thus henselian. This subfield is also truncation closed by Corollary 2.9. Thus replacing E by $E(t^{\gamma} : \gamma \in \mathbb{Q}\Delta)$ we arrange that the value group Δ of E is divisible. By passing to the (truncation closed) subfield $\bigcup_{n=1}^{\infty} E^{1/p^n}$ of K, we can also assume that E is perfect. Note that the properties we arranged to hold for E are preserved by any truncation closed field intermediate between E and E^{a}. Thus it suffices to show that, under the present assumptions on E, if also $E \neq E^{\mathrm{a}}$, then there is a truncation closed field F intermediate between E and E^{a} with $E \neq F$. So assume $E \neq E^{\mathrm{a}}$. By familiar extension theory of valued fields, if F is a field extension of E of finite degree, then $[F : E] = p^n$ for some n. Thus well-known Galois theory yields $a \in E$ such that the Artin-Schreier polynomial $T^p - T - a$ has no zero in E. Now $a = a_1 + a_2$ with $a_1 = a|_0$ and $a_2 \in \mathcal{O}_E$. By henselianity, $T^p - T - a_2$ does have a zero in E, and so $T^p - T - a_1$ has no zero in E. Replacing a by a_1 we may therefore assume that $\operatorname{supp} a < 0$. Among the $a \in E$ with $\operatorname{supp} a < 0$ for which $T^p - T - a$ has no zero in E we pick one with minimal ordinal $o(a)$. For this a the hypotheses of Lemma 5.3 are satisfied, and so we obtain a proper extension $F := E(\omega(a)) \subseteq E^{\mathrm{a}}$ of E which is truncation closed by Corollary 2.9. □

Counterexamples. Let $\operatorname{char}(\boldsymbol{k}) = p > 0$ and let $\Gamma \neq \{0\}$ be p-divisible. We shall indicate a henselian truncation closed subfield $E \supseteq \boldsymbol{k}$ of K with an algebraic field extension $F \subseteq K$ that is not truncation closed. (This kind of behaviour is impossible when the residue characteristic is 0, by Theorem 5.1.)

Take $\alpha < 0$, set $\Delta := \mathbb{Z}\alpha$ and $E := \boldsymbol{k}((t^{\Delta})) \subseteq K$. Then E is a henselian truncation closed subfield of K. Set $a = t^{\alpha}$. Then $E(\omega(a)) \subseteq K$ is a cyclic Galois extension of degree p of E, but is not truncation closed: the leading term $t^{\alpha/p}$ of $\omega(a)$ is purely inseparable over E, and so does not lie in $E(\omega(a))$.

This E is not perfect, but now we give an example where E is perfect. For this we assume also that Γ has rank > 1. Take $\alpha, \beta < 0$ such that $\alpha < n\beta$ for all n. Set

$$\Delta_n := \frac{\mathbb{Z}\alpha + \mathbb{Z}\beta}{p^n} \subseteq \Gamma, \qquad E := \bigcup_{n=0}^{\infty} \boldsymbol{k}((t^{\Delta_n})) \subseteq K.$$

Then E is a perfect henselian truncation closed subfield of K. Set $a = t^{\alpha}$, $b = t^{\beta}$. Then $a, b, a+b \in E$, but $\omega(a), \omega(b), \omega(a+b) \notin E$. All proper truncations of $\omega(a)$ and $\omega(b)$ are in E, so the subfields $E(\omega(a))$ and $E(\omega(b))$ of K are truncation closed (and are cyclic Galois extensions of degree p of E). Also, $E(\omega(a)) \neq E(\omega(b))$ by familiar facts about Artin-Schreier extensions. Now $\omega(a+b) = \omega(a) + \omega(b)$ has $\omega(a)$ as a proper truncation. We claim that the subfield $F := E(\omega(a+b))$ of K is not truncation closed because it does not contain $\omega(a)$: if it did, then it would contain also $\omega(b)$, and so F would have degree $> p$ over E, which is not the case.

The twisted case. Let $L = \boldsymbol{k}((t^{\Gamma}; \mathfrak{c}))$ be a twisted Hahn field. Theorems 4.3 and 5.1 go through for L instead of K and subfields E of L with $E \supseteq \boldsymbol{k}$, with the same proofs except for trivial changes. (Of course, "strongly truncation closed" is equivalent to "truncation closed" for such E.)

6. Kaplansky embedding with truncation

Let F be a field with (surjective) valuation $v_F : F^\times \to \Gamma$ and residue field $\boldsymbol{k}$. By a *copy* of Γ in F we mean a subgroup of $F^\times$ that is mapped bijectively onto Γ by v_F. By a *copy* of $\boldsymbol{k}$ in F we mean a subfield of F that is contained in $\mathcal{O}_F$ and is mapped onto $\boldsymbol{k}$ by the residue map $\mathcal{O}_F \to \boldsymbol{k}$. A *truncation closed copy* of F is a field embedding $\phi : F \to \boldsymbol{k}((t^\Gamma))$ such that for all $a \in F$,

(1) if $a \in \mathcal{O}_F$, then $\big(\text{residue class of } a\big) = \big(\text{constant term } \phi(a)_0 \text{ of } \phi(a)\big)$;
(2) $v_F(a) = v(\phi(a))$ (so ϕ is a valued field embedding);
(3) $\phi(F)$ is truncation closed in $\boldsymbol{k}((t^\Gamma))$.

Given such a truncation closed copy $\phi : F \to \boldsymbol{k}((t^\Gamma))$, it follows from (1) and (3) that $\boldsymbol{k} \subseteq \phi(F)$, so $\phi^{-1}(\boldsymbol{k})$ is a copy of $\boldsymbol{k}$ in F, and then by (2) we get $t^\Gamma \subseteq \phi(F)$, so $\phi^{-1}(t^\Gamma)$ is a copy of Γ in F.

Thus a necessary condition for the existence of a truncation closed copy of F is that the residue field $\boldsymbol{k}$ as well as the value group Γ have copies in F. It is well-known that if F is henselian and $\operatorname{char} \boldsymbol{k} = 0$, then $\boldsymbol{k}$ has a copy in F. It is also easy to see that if Γ is free as an abelian group, or $F^\times$ has a divisible subgroup that is mapped onto Γ by v_F, then Γ has a copy in F.

Theorem 6.1. *Suppose F is henselian and* $\operatorname{char} \boldsymbol{k} = 0$*. Then:*

$$F \text{ has a truncation closed copy} \iff \Gamma \text{ has a copy in } F.$$

As we saw, the direction $\Rightarrow$ holds even without the assumptions on F in this theorem. The converse is a special case of the more precise Proposition 6.3 below. The main argument is in the proof of the next lemma, and resembles one in [9]. We now return to the setting of an ambient Hahn field $K = \boldsymbol{k}((t^\Gamma))$.

Lemma 6.2. *Assume* $\operatorname{char}(\boldsymbol{k}) = 0$*. Let $E \supseteq \boldsymbol{k}$ be a truncation closed subfield of K and F an immediate henselian valued field extension of E. Then the inclusion $E \to K$ extends to a valued field embedding $F \to K$ with truncation closed image.*

Proof. By the universal property of the henselization of E and by Theorem 1.2 we can arrange that E is henselian. If $F = E$ we are done, so assume $F \neq E$. Take any $f \in F \setminus E$, and take a pseudocauchy sequence (e_λ) in E that pseudoconverges to f and has no pseudo limit in E. Then (e_λ) is of transcendental type and has a pseudolimit in K. Take such a pseudolimit $g \in K$ for which $o(g)$ is minimal.It is routine to check that then all proper truncations of g are in E, and so $E(g)$ is truncation closed by Corollary 2.9. The inclusion $E \to K$ extends to a valued field embedding $E(f) \to K$ sending f to g, and so has truncation closed image $E(g)$. Now iterate this extension procedure till all of F is embedded into K. □

Let F be a henselian valued field of residue characteristic 0. Then the residue field of F has a copy in F, and we assume given such a copy $\mathfrak{r}$. We also assume given a copy G of the value group of F in F. We make G into ordered group such that the valuation of F restricts to an ordered group isomorphism from G onto the value group of F. In this situation we have:

Proposition 6.3. *Let $i : \mathfrak{r} \to \boldsymbol{k}$ be a field embedding and $j : G \to \Gamma$ an ordered group embedding. Then there is a valued field embedding $\phi : F \to K$ such that*

(1) *ϕ extends i and $\phi(g) = t^{jg}$ for all $g \in G$;*
(2) *$\phi(F)$ is strongly truncation closed in K.*

Proof. Replacing K by its Hahn subfield $i(\mathfrak{r})((t^{jG}))$, we arrange that i and j are isomorphisms. By Corollary 2.9, $E := \boldsymbol{k}(t^\Gamma)$ is a truncation closed subfield of K. Identifying the valued subfield $\mathfrak{r}(G)$ of F with E via the valued field isomorphism $\mathfrak{r}(G) \to E$ that extends i and sends each $g \in G$ to t^{jg}, we are in the situation of Lemma 6.2. Applying this lemma gives the desired result. □

Kaplansky's embedding theorem also includes the positive residue characteristic case provided the so-called Kaplansky conditions are satisfied. Extensions of the above to this situation can be found in [5].

7. Stability under certain transcendental adjunctions

Recall from the Introduction that for each $n \geqslant 1$ we are given a subset $\mathcal{F}_n$ of $\boldsymbol{k}[[X_1,\dots,X_n]]$ such that the subring $\boldsymbol{k}[X_1,\dots,X_n,\mathcal{F}_n]$ of $\boldsymbol{k}[[X_1,\dots,X_n]]$ is closed under $\partial/\partial X_i$ for $i = 1,\dots,n$, and that $\mathcal{F} = (\mathcal{F}_n)$. Then each subfield E of K extends to its $\mathcal{F}$-closure $\mathcal{F}(E) \subseteq K$.

Proof of Theorem 1.3. We assume $\mathrm{char}(\boldsymbol{k}) = 0$ and $E \supseteq \boldsymbol{k}$ is a truncation closed subfield of K. Our task is to show that the $\mathcal{F}$-closure $\mathcal{F}(E)$ is truncation closed.

Let F be a truncation closed subfield of $\mathcal{F}(E)$ containing E such that $F \neq \mathcal{F}(E)$; by invoking Corollary 2.9 and Zorn it suffices to show that some element of $\mathcal{F}(E)\setminus F$ has all its proper truncations in F. Let $n \geqslant 1$ be minimal such that there are $a_1,\dots,a_n \in \mathfrak{m}\cap F$ and $f \in \mathcal{F}_n$ with with $f(a_1,\dots,a_n) \notin F$. To simplify notation, assume $n = 2$; the general case is similar. With the lexicographic ordering on the relevant set of pairs of ordinals, we can take $a, b \in \mathfrak{m}\cap F$ with minimal $(o(a), o(b))$ such that $f(a,b) \notin F$ for some $f \in \mathcal{F}_2$. Fix such an f. It is enough to show that all proper truncations of $f(a,b)$ lie in F.

Let c be a proper truncation of $f(a,b)$. We have $a \neq 0$ or $b \neq 0$, so we can take $\gamma \in \operatorname{supp} a \cup \operatorname{supp} b$ and a positive integer N such that $N\gamma > \operatorname{supp} c$. Assume $\gamma \in \operatorname{supp} b$. (The other case is similar.) Then $b = b_0 + \varepsilon$ with $b_0, \varepsilon \in F$ and $\operatorname{supp} b_0 < \gamma$, $v(\varepsilon) = \gamma$. (This allows $b_0 = 0$.) Consider the Taylor expansion

$$f(a,b) \;=\; f(a, b_0+\varepsilon) \;=\; \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial X_2^n}(a,b_0)\frac{\varepsilon^n}{n!}.$$

We have $(\partial^n f/\partial X_2^n)(a,b_0) \in \mathcal{O}$ for all n, so for $n \geqslant N$,

$$v\Big(\frac{\partial^n f}{\partial X_2^n}(a,b_0)\frac{\varepsilon^n}{n!}\Big) \;\geqslant\; n\gamma \;\geqslant\; N\gamma \; > \operatorname{supp} c.$$

Thus c is a truncation of d, where

$$d \;:=\; \sum_{n<N} \frac{\partial^n f}{\partial X_2^n}(a,b_0)\frac{\varepsilon^n}{n!}.$$

Since $o(b_0) < o(b)$, the minimality property of $(o(a), o(b))$ guarantees

$$(\partial^n f/\partial X_2^n)(a,b_0) \in F$$

for $n < N$, so $d \in F$. Since F is truncation closed, this gives $c \in F$. This concludes the proof of Theorem 1.3.

8. Final Comments

Truncation closure seems to have been considered first in [9] for subfields of Hahn fields $\mathbb{R}((t^\Gamma))$ with divisible Γ, in connection with showing that every real closed field R has an *integral part*, that is, R has a subring Z such that for each $a \in R$ there is a unique $z \in Z$ with $z \leqslant a < z+1$. (This is an issue that comes up in studying certain weak fragments of Peano arithmetic.) Indeed, much of Section 2 of the present paper is in some form in [8] and [9]. So is part of Theorem 5.1.

The referee pointed out that Fornasiero [5] contains Theorems 1.1 and 1.2. (The proof of part (1) of 1.2 in that paper is different from ours.) Moreover, Fornasiero's paper deals in more detail than we do with truncation closed subfields of *twisted* Hahn fields, and also considers truncation closed Kaplansky embeddings in *positive* characteristic.

Theorem 1.3 and the contents of Section 7 stem from [1], where the setting is a bit more restricted, and the $\mathcal{F}$-closure has a slightly different meaning. There it is used to show that the Gamma function on $(0, \infty)$ is not definable in the o-minimal expansion $\mathbb{R}_{\text{an,exp}}$ of the real field.

Truncation closed subsets of Hahn fields also show up as *initial* subsets of Conway's Field **No** of surreal numbers; see [2]. One benefit of working with (strongly) truncation closed subfields of Hahn fields is that it suggests new inductive procedures in establishing embedding theorems. We saw some of this in Section 6 above. For more, see [10], [5], and [4].

My thanks to Matthias Aschenbrenner for bringing [11] to my attention.

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